

Connection-state approach to pre- and post-selected measurements

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We introduce the concept of connection states, i.e., operators that describe posterior ensembles, post-selected according to different outcomes of a quantum measurement. Connection states allow one to retrodict results of any weak pre- and the post-selected measurements, in the same manner as density matrices allow one to predict the results of conventional quantum measurements. Connection-state operators are generally non-Hermitian, i.e., they differ drastically from the density matrices. This is shown to be a direct consequence of the non-classical nature of quantum mechanics. The non-Hermiticity of connection-state operators explains the unusual character of weak values. We show that connection-state operators can be determined experimentally.

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Quantum mechanics is intrinsically stochastic. Measurements of a physical quantity for an ensemble of identical quantum systems prepared in the same state (the so-called “preselected” ensemble) yield different outcomes for different individual systems. The quantum state, described generally by the density matrix ρ , contains all the information on a quantum system available for an observer at a given time t_0 and thus allows one to predict the probabilities of the outcomes for any future measurement performed on the system.

A measurement performed at a time t_1 ($t_1 > t_0$) provides new information about the properties of the quantum system in the interval (t_0, t_1) . This information allows one to divide the preselected ensemble into pre- and post-selected (PPS) ensembles, i.e., sub-ensembles corresponding to different measurement outcomes. Such ensembles can be probed by measurements performed at intermediate times $t \in (t_0, t_1)$, i.e., by PPS measurements [1]. The results of PPS measurements obviously depend not only on ρ but also on the final-measurement outcome. Therefore, the following question arises: Can one describe a PPS ensemble similarly to a pre-selected ensemble, i.e., by some operator, which can be used to predict retroactively (retrodict) the full statistics of any PPS measurement? Below it is shown that indeed, under certain conditions, such an operator exists. It is called here the “connection state”, since it describes quantum systems between the preparation of the initial state and the measurement and thus connects these two stages of evolution. The connection state is useful for retrodiction, in contrast to the conventional state operator (density matrix) used for prediction. Remarkably, the connection-state operator is generally non-Hermitian. Below this property is shown to be a direct consequence of the non-classical nature of quantum mechanics.

PPS measurements [1], especially weak PPS measurements [2] and the resulting weak values of observables, were found useful in a multitude of interesting applica-

tions [3, 4], including foundations of quantum mechanics [5, 6], high-precision metrology [7], and measuring wavefunctions [8]. Most of these applications involve unusual weak values, which can be complex numbers with unbounded magnitudes and real parts lying far outside the range of the eigenvalues of the observable.

The physical meaning of weak values is not yet completely understood and is a subject of controversy [9]. This is an impediment to further progress in the field of PPS measurements. Our approach sheds new light on PPS measurements. In particular, below we show that the unusual character of weak values is explained by the non-Hermiticity of *connection-state operators* and hence is a direct consequence of the non-classical nature of quantum mechanics.

Quantum measurements.— We begin with a very brief overview of quantum measurements. Consider first ideal (or strong) quantum measurements. Let ρ be the density matrix describing the state of the quantum system. The operator of any physical quantity A has the spectral expansion

$$A = \sum_i a_i \Pi_i. \quad (1)$$

Here a_i ($a_i \neq a_j \forall i \neq j$) are the eigenvalues of A , and Π_i are projection operators. Then, according to the projection postulate [10, 11], an ideal (strong) measurement of A yields an eigenvalue a_i with probability

$$P_i = \text{Tr}(\rho \Pi_i), \quad (2)$$

while leaving the system in the state $\Pi_i \rho \Pi_i / \text{Tr}(\rho \Pi_i)$. Furthermore, for general measurements, the probability of the i th measurement outcome is given by [12]

$$P_i = \text{Tr}(\rho E_i). \quad (3)$$

Here E_i constitute the positive-operator valued measure (POVM) satisfying $\sum_i E_i = I$, where I is the unity operator.

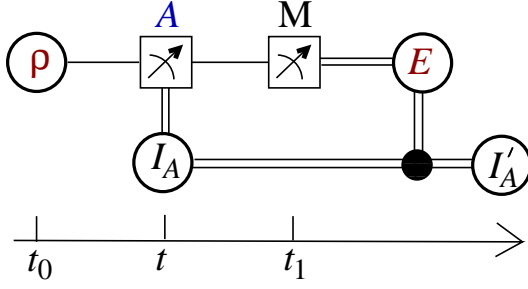


FIG. 1. Schematic diagram of PPS measurements. A measurement of an observable A in a state ρ provides information I_A (a conventional quantum measurement). In a PPS measurement, this information is conditioned on a result of a subsequent measurement M . Here ordinary (double) lines carry quantum (classical) information.

Consider now PPS measurements (see Fig. 1). We begin with ideal (or strong) PPS measurements [1]. In the general case, when the initial state ρ can be a mixed state and the post-selection is made by an outcome of a general measurement with a POVM operator E , it is easy to show that the probability to observe an eigenvalue a_i of A in a strong PPS measurement is given by [4]

$$P_{i|E} = \text{Tr}(E \Pi_i \rho \Pi_i) / \sum_j \text{Tr}(E \Pi_j \rho \Pi_j). \quad (4)$$

This formula is an extension of the results obtained in Refs. [1, 3], and it reduces to them when the initial state is pure and the post-selection measurement is ideal. Studies of strong PPS measurements provided some interesting results, such as the 3-box quantum paradox [13, 14] and the time-symmetry relation [1, 3, 4, 13].

An alternative to strong PPS measurements are weak PPS measurements [2]. Weak PPS measurements of a quantity A yield the so-called weak value of A , which in the general case is given by [15, 16]

$$A_w = \text{Tr}(E A \rho) / \text{Tr}(E \rho). \quad (5)$$

In the special case when the initial state is pure, $\rho = |\psi\rangle\langle\psi|$, and the final measurement is ideal with E being a rank-1 projector, $E = |\phi\rangle\langle\phi|$, the weak value is [2] $A_w = A_{\phi\psi} / \langle\phi|\psi\rangle$. In this case, it is common to say that “the system is post-selected in the state $|\phi\rangle$ ”, since the final state is $|\phi\rangle$. Note, however, that generally the final state is not uniquely determined by E and is dependent on ρ as well. What is more important, *the final state is irrelevant for PPS measurements*, since they are not affected by the evolution of the system after the post-selection. Still, we will use the common term “a pure post-selected state” to refer to the cases where E is a rank-1 projector.

PPS measurements generally significantly change the initial state ρ of a quantum system, and as a result they

may affect PPS ensembles [17]. In particular, for strong PPS measurements this effect is seen from the fact that the post-selection probability, given by the denominator in Eq. (4), depends explicitly on A . Below we focus mainly on the cases where the dependence of PPS ensembles on the intermediate measurements is negligibly small. In this case, PPS ensembles are very close to *posterior ensembles*, a special case of PPS ensembles where no intermediate (PPS) measurements are performed.

Connection states.— Weak PPS measurements (at least, in the linear-response regime [4]) have the important property that they do not appreciably disturb the state of the quantum system. Hence they probe unperturbed PPS ensembles (i.e., posterior ensembles). Let us show that weak PPS measurements are similar in a sense to conventional (preselected only) measurements. For strong conventional measurements, the projection postulate implies that the measured expectation value of the quantity A , $\bar{A} = \sum_i a_i P_i$, is given by [cf. Eqs. (1) and (2)]

$$\bar{A} = \text{Tr}(A \rho). \quad (6)$$

This expectation value is obtained also in weak conventional measurements [2]. The weak value (5) can be recast in a form similar to Eq. (6) (on using the invariance of the trace under cyclic permutations),

$$A_w = \text{Tr}(A w), \quad (7)$$

where

$$w = \frac{\rho E}{\text{Tr}(\rho E)}. \quad (8)$$

In the special case of pure pre- and post-selected states (i.e., $\rho = |\psi\rangle\langle\psi|$ and $E = |\phi\rangle\langle\phi|$), Eq. (8) becomes $w = |\psi\rangle\langle\phi| / \langle\phi|\psi\rangle$.

The quantity w can be called the *connection-state operator*. It determines the results of weak PPS measurements in the same way as the quantum state determines the results of conventional weak measurements [cf. Eqs. (6) and (7)]. Therefore the connection-state operator generalizes the concept of the quantum state operator (density matrix). In particular, the connection state reduces to the quantum state of the system, $w = \rho$, when E is equal or proportional to the unity operator. This could be expected, since in this case the post-selection measurement does not yield any information or is not performed at all, and thus the posterior ensemble reduces to a preselected ensemble.

The operators ρE and w have simple physical meanings. Indeed, we note that ρ and E are quantum counterparts of the prior probability distribution and the conditional probability of the measurement outcome, respectively [18]. This implies that ρE is a quantum counterpart of the joint probability distribution, and, hence,

the connection-state operator w in Eq. (8) is a quantum counterpart of the classical posterior probability distribution (i.e., the probability distribution conditioned by a measurement outcome). In this interpretation, Eq. (8) is a quantum analog of Bayes' theorem [recall that $\text{Tr}(\rho E)$ is the probability of the measurement outcome, cf. Eq. (3)]. Obviously, connection states are normalized to one, $\text{Tr} w = 1$. Note, however, an important distinction of the present situation from the classical case. A peculiarity of quantum probability theory [18, 19] is that ρ and E generally do not commute. As a result, *connection-state operators are generally non-Hermitian*, i.e., they have drastically different properties from conventional density matrices.

This explains why weak values are generally unusual. Indeed, the connection-state operator can be written in the form,

$$w = w' + iw'', \quad w' = \frac{\rho E + E \rho}{2\text{Tr}(\rho E)}, \quad w'' = \frac{[\rho, E]}{2i\text{Tr}(\rho E)}. \quad (9)$$

Here w' and w'' are Hermitian operators, which determine the real and imaginary parts of the weak value, $\text{Re} A_w = \text{Tr}(Aw')$ and $\text{Im} A_w = \text{Tr}(Aw'')$ [20]. Note that $\text{Tr} w' = 1$ and $\text{Tr} w'' = 0$. The eigenvalues of w' and w'' can be positive or negative. As a result, the magnitudes of these eigenvalues are not restricted by the above condition $\text{Tr} w = 1$ and thus can be arbitrarily large. This can be shown explicitly by considering the connection-state norm $\|w\|$ given by the square root of the maximum eigenvalue of $w^\dagger w$ (or, equivalently, ww^\dagger). In particular, it is easy to see that in the case of pure pre- and post-selected states, $\|w\| = |\langle \phi | \psi \rangle|^{-1}$, tending to infinity for $\langle \phi | \psi \rangle \rightarrow 0$. This gives rise to unbounded weak values.

Consider the POVM $\{E_i\}$ for a general measurement and the corresponding connection states $w_i = \rho E_i / P_i$. The normalization condition $\sum_i E_i = I$ implies the following sum rule for the connection states,

$$\sum_i P_i w_i = \rho, \quad (10)$$

where P_i is the probability of the outcome i , Eq. (3). In turn, Eq. (10) implies a sum rule for the weak values $A_{wi} = \text{Tr}(E_i A \rho) / \text{Tr}(E_i \rho)$ of a quantity A . Indeed, multiplying both sides of Eq. (10) by A and taking the trace yields $\sum_i P_i A_{wi} = \bar{A}$. Special cases of this sum rule were obtained in Refs. [4, 21].

Connection states can be classified into usual and unusual, depending on whether they allow for unusual weak values. A connection state, Eq. (8), is usual, if and only if ρ and E commute. Indeed, when ρ and E commute, w is a Hermitian, positive operator, similar to a density matrix, and, as a result, any weak value is usual, i.e., a real number inside the range of the eigenvalues of the observable, just as the expectation value [cf. Eqs. (6) and

(7)]. On the other hand, when ρ and E do not commute, the connection state is unusual, i.e., then there always exists an observable with an unusual weak value. Indeed, then w is non-Hermitian and hence $w'' \neq 0$. In this case, there exists a Hermitian operator A such that $\text{Tr}(Aw'') = \text{Im} A_w \neq 0$; hence A possesses a complex (i.e., unusual) weak value. In particular, for pure pre- and post-selected states $|\psi\rangle$ and $|\phi\rangle$, connection states are always unusual, except for the trivial case $|\psi\rangle = |\phi\rangle$ where the posterior and preselected ensembles coincide.

The unusual character of the connection states in the case where ρ and E do not commute can be explained by the fact that in this case the final measurement probes an aspect of the quantum system which is *incompatible* with the aspect described by the initial state ρ . In this case, according to the postulates of quantum mechanics, it is generally impossible to use the information provided by the final measurement to improve the knowledge on a certain observable obtained by a measurement in the state ρ . However, if the connection state were usual (i.e., a positive operator), it would provide an improved knowledge on any observable, which is forbidden.

Still, even when the connection-state operator is unusual (i.e., non-Hermitian), there exist observables with usual weak values. It is interesting that, for an arbitrary connection state, weak values are always usual whenever A commutes with either ρ or E [4].

Connection-state operators can be determined experimentally. In particular, quantum tomography of connection states can be performed with the help of weak PPS measurements, similarly to tomography of quantum states [12, 22]. Indeed, a connection state can be written in the form $w = I/d + \sum_{i=1}^{d^2-1} \alpha_i A_i$, where $\{A_i\}$ is a set of linearly-independent Hermitian operators with zero trace, d is the dimension of the Hilbert space of the quantum system, and α_i are complex coefficients. On multiplying the above expression for w by A_j , taking the trace, and using Eq. (7), we obtain the equations, $\sum_{i=1}^{d^2-1} a_{ji} \alpha_i = (A_j)_w$, where $a_{ji} = \text{Tr}(A_j A_i)$ and $(A_j)_w$ are weak values that can be measured. Solving these equations yields α_i and hence the connection state w . Namely, $\alpha_i = \sum_{j=1}^{d^2-1} (a^{-1})_{ij} (A_j)_w$, where a is the matrix with the elements a_{ji} .

Evolution of connection states.— Until now, we neglected the effects of the Hamiltonian of the quantum system. Consider now the time evolution due to the system Hamiltonian $H(t)$. We assume that the initial state is prepared at time t_0 and the post-selection measurement is performed at time $t_1 > t_0$. Moreover, we assume that a PPS measurement is performed using the von-Neumann scheme [2, 10], so that the system and meter are correlated impulsively at time t ($t_0 < t < t_1$). Then it is easy to show [4] that in the general formulas for PPS measurements of arbitrary strength, time evolution is taken into account

by the substitutions $\rho \rightarrow \rho(t) = U(t, t_0)\rho U^\dagger(t, t_0)$ and $E \rightarrow E(t_1, t) = U^\dagger(t_1, t)EU(t_1, t)$. Here $U(t, t') = T \exp[-i \int_{t'}^t d\tau H(\tau)/\hbar]$, where T is the chronological operator. The POVM operator in the Heisenberg picture, $E(t_1, t)$, satisfies the backward Heisenberg equation [23], $i\hbar dE/dt = [H(t), E]$, which should be solved backward in time with the final condition $E(t_1, t_1) = E$. $E(t_1, t)$ is a quantum counterpart of the probability of a measurement outcome at t_1 given the system state at t . This can be seen from the fact that the (unconditional) probability of the measurement outcome $P = \text{Tr}[\rho(t_1)E]$ can be recast as $P = \text{Tr}[\rho(t)E(t_1, t)] = \text{Tr}[\rho E(t_1, t_0)]$.

Inserting the above substitutions into Eq. (8) yields the time-dependent connection state,

$$w(t) = \rho(t)E(t_1, t)/P. \quad (11)$$

In view of the above interpretation of $E(t_1, t)$, this time dependence of $w(t)$ is formally the same as for a classical posterior probability distribution. Equation (11) can be called the *time-dependent quantum Bayes formula*. More explicitly,

$$w(t) = U(t, t_0)\rho U^\dagger(t_1, t_0)EU(t_1, t)/P. \quad (12)$$

In the case of pure pre- and post-selected states, Eq. (12) yields [24] $w(t) = U(t, t_0)|\psi\rangle\langle\phi|U(t_1, t)/\langle\phi|U(t_1, t_0)|\psi\rangle$.

The weak value in the presence of free evolution of the quantum system is given by $A_w = \text{Tr}[Aw(t)]$ or, in view of Eq. (12), by

$$A_w = \frac{\text{Tr}[AU(t, t_0)\rho U^\dagger(t_1, t_0)EU(t_1, t)]}{\text{Tr}[\rho(t_1)E]}. \quad (13)$$

In addition to the above ‘‘Schrödinger picture’’, one can use also two other representations. Equation (13) can be recast in the ‘‘forward Heisenberg picture’’, $A_w = \text{Tr}[A(t, t_0)w(t_0)]$, where $A(t, t_0) = U^\dagger(t, t_0)AU(t, t_0)$, or in the ‘‘backward Heisenberg picture’’, $A_w = \text{Tr}[A(t_1, t)w(t_1)]$. In these pictures only the observable evolves, whereas the connection state is constant in time, being equal to its initial (final) state $w(t_0)$ [$w(t_1)$]. These pictures are extensions of the Heisenberg picture in quantum mechanics to the case of connection states.

Discussion.— It is of interest to compare the connection state w in Eq. (8) with the posterior conditional state obtained due to an ‘‘efficient measurement’’ [18],

$$\rho' = M\rho M^\dagger/\text{Tr}(\rho E), \quad (14)$$

where M is the measurement operator satisfying $M^\dagger M = E$. Along with Eq. (8), the equality (14) can be also considered as a quantum counterpart of Bayes’ theorem [25]. Actually, when the measurement is minimally disturbing ($M = \sqrt{E}$) and E commutes with ρ , Eqs. (8) and (14) coincide. However, generally w and ρ' are very different. Thus, we obtain that generally a posterior ensemble is characterized by two quantities: the conventional

state ρ' used for predictions in the future ($t > t_1$) and the connection state w used for retrodiction in the past ($t_0 < t < t_1$).

Let us compare the present connection-state formalism with the two-state vector formalism (TSVF) developed by Aharonov et al. [1, 13, 26]. Our formalism significantly differs from the TSVF, at least, in two aspects. First, our formalism uses one operator rather than two vectors, and, second, in contrast to the TSVF which considers the special case of pure pre- and post-selected states, our formalism describes the general case with arbitrary initial states and post-selection measurements.

In summary, we introduced the notion of connection state operator that describes quantum systems in a posterior ensemble produced by a quantum measurement. A connection state is a non-Hermitian operator that is a direct extension of the density matrix. We have discussed the physical meaning of connection states and their relation to PPS measurements and weak values. We have shown that the unusual character of weak values is a direct consequence of the non-Hermiticity of connection states, which in turn results from the non-classical nature of quantum randomness. The present approach opens new venues in quantum mechanics. In particular, here it is shown that a broad class of non-Hermitian operators are experimentally accessible quantities with a clear physical meaning. The present approach can be useful in all applications of PPS measurements, including quantum information processing.

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